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TWO-DIMENSIONAL PROBLEM OF THE MOTION OF A SNOW AVALANCHE ALONG A SLOPE WITH SMOOTHLY CHANGING PROPERTIES

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To describe the motion of an avalanche we use "hydraulic" models, one version of which for a one-dimensional motion was proposed in [1]. An asymptotic solution as $t \to \infty$ was constructed in [2] for the equations proposed in [1] for the case of a slope of constant steepness with a uniform snow cover. Below we investigate the asymptotic behavior of the solution of a two-dimensional problem of the motion of a snow avalanche along a slope of varying steepness, on which snow with varying properties lies. It is assumed that the typical linear scale of variation of these quantities is sufficiently large.

1. Statement of the problem. The equations of two-dimensional motion of a snow avalanche, analogous to those proposed in [1] for the one-dimensional case, are written in the form

$$dh / dt + h \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{g}{2h} \operatorname{grad}(h^2 \cos\psi) + \mathbf{e} g \sin\psi - F(v, h, x, y)\mathbf{v} \tag{1.2}$$

Here v is the snow's velocity averaged over the thickness, h is the thickness of the moving snow layer, ψ is the angle between the horizontal plane and the tangent plane to the slope at a point being considered, $\mathbf{e}$ is a vector lying in the tangent plane and specifying the direction of steepest descent, $F(v, h, x, y)\,\mathbf{v}$ is the force of friction referred to unit mass, t is the time, x, y are certain coordinates on the slope's surface, and the operations div and grad are taken on the slope's surface. An expression for the force of friction was used in [1, 2], which can be written as

$$F(v, h, x, y)\,\mathbf{v} = \mu g \cos\psi \frac{\mathbf{v}}{|v|} + k\frac{|v|\,\mathbf{v}}{h} \tag{1.3}$$

Here μ is the coefficient of "dry" friction, k is the coefficient of hydraulic friction.

Equations (1.1), (1.2) apply only in the region where the snow moves. We shall assume that the relations

$$h(w_n - v_n) = h_0 w_n \tag{1.4}$$
$$\rho h_0 w_n \mathbf{v} = {}^1/_2 \rho g h^2 \cos\psi \mathbf{n} - \boldsymbol{\sigma}^* h$$

analogous to those used in [1, 2] and expressing the conservation of mass and momentum, are fulfilled at the leading front of the avalanche, i.e. at the boundary of the snow at rest and in motion. Here w_n is the velocity of motion of the front (along the normal), v_n is the velocity of the snow behind the front, ρ is the density of the moving snow, which we assume to equal the density of the snow lying ahead of the avaloanche, $\boldsymbol{\sigma}^*$ is the critical stress at which the structure of the moving snow breaks down.

The surface of discontinuity described by relations (1.4) we shall call the break-down front. We assume that at the break-down front

$$\boldsymbol{\sigma}^* = \sigma^* \mathbf{n} \tag{1.5}$$

Then the second equation in (1.4) yields

$$\rho h_0 w_n v_n = {}^1/_2\, \rho g h^2 \cos\psi - \sigma^* h_0, \quad v_\tau = 0$$

Under natural assumptions on the snow's properties and on its break-down process, the relation

$$\sigma^* h_0 \geqslant {}^1/_2\, \rho g h_0^2 \cos\psi \tag{1.6}$$

should be fulfilled; the equality holds only when the snow ahead of the front behaves like a liquid. Surfaces of discontinuity can be formed also within the avalanche as it moves. We assume that the very same conditions as at hydraulic discontinuities

$$h^+(w_n - v_n) = h^-(w_n - v_n^-) \tag{1.7}$$
$$h^+(w_n - v_n^+)\,v_n^+ - h^-(w_n - v_n^-)\,v_n^- =$$
$$g\left(\frac{h^{+2}}{2} - \frac{h^{-2}}{2}\right)\cos\psi \qquad v_\tau^+ = v_\tau^-$$

are fulfilled on these surfaces. Here the minus and plus indices denote, as usual, that the corresponding quantity is taken ahead of or behind the surface of discontinuity.

By L^* we denote the distance at which the quantities characterizing the properties of the slope and of the snow cover change by a substantial part of their own magnitude. By L and T we denote the typical scale and the characteristic time of variation of the solution, where, as usual, we can assume that $T = L / v^*$, and where v^* is the typical velocity of the avalanche. It is obvious that $L \leqslant L^*$. The ratio of the left-hand side of Eq. (1.2) to the force of friction is of the order of the quantity v^* / FL, while the ratio of the first term on the right-hand side to the force of friction is of the order of $hg\cos\psi / Fv^* L$. The differential terms in Eq. (1.2) can be neglected for sufficiently large values of L, namely, for

$$L \gg \max\left(\frac{v^*}{F},\ \frac{gh\cos\psi}{Fv^*}\right) \tag{1.8}$$

The equations obtained in this manner are said to be simplified. They describe the large-scale motions.

When $F(v, h, x, y)$ is expressed by formula (1.3) the condition under which we can neglect the differential terms can be written as

$$L \gg \max\left(\frac{h^*}{k},\ \frac{h^*}{\operatorname{tg}\psi-\mu}\right) \tag{1.9}$$

where h^* is the typical depth of the snow in the avalanche, equal in order of magnitude to the initial depth of the snow. The neglecting of the differential terms in Eq.(1.2) when constructing the asymptotic solution had been done earlier in [2] for the study of snow motion and in [3] for the study of the motions of a liquid layer.

The use of a simplified system to describe the motion on long smoothly varying slopes is feasible under the condition that the motion with slowly varying parameters is stable, because otherwise there can appear perturbations with a characteristic linear size not satisfying condition (1.8). However, if we are interested not in the fine-scale perturbations but only in the average characteristics of the motion, then (as is done in the study of turbulent motions) we can use the same Eqs. (1.1), (1.2) with another force of friction $F\mathbf{v}$ depending on average v and h. Obviously, condition (1.8) is fulfilled for such an average motion along a slope with large L^*. We shall assume that condition (1.8) is fulfilled over a sufficiently large time after the start of the motion almost over the whole region occupied by the moving avalanche, except for a narrow region directly abutting the leading front of the avalanche (the break-down front) and, possibly, certain narrow regions within the avalanche. From the large-scale point of view these narrow regions of rapid variations of parameters of the moving snow can be replaced by surfaces of discontinuity.

2. General solution of the simplified system. Solving Eq. (1.2) with the differential terms discarded relative to $\mathbf{v}$, we have

$$\mathbf{v} = V(h, x, y)\,\mathbf{e} \tag{2.1}$$

Thus, the velocity is directed along the line of steepest descent. Therefore, the solution for the whole slope can be obtained by dividing it up into narrow strips by lines of steepest descent and examining the motion inside each strip.

Farther on, by x we denote the distance along some strip and by $S(x)$ the width of this strip. Then from Eqs. (1.1) and (2.1) we obtain

$$\frac{\partial Sh}{\partial t} + \frac{\partial}{\partial x}[ShV(h, x)] = 0 \tag{2.2}$$

The equations for the characteristics of this equation have the form

$$\frac{dx}{dt} = \frac{\partial}{\partial h}[hV(h, x)] \tag{2.3}$$
$$\frac{dh}{dt} = -h\left(\frac{\partial V}{\partial x} + V\frac{d\ln S}{dx}\right)$$

If we take expression (1.3) for the force of friction in the moving avalanche, then relation (2.1), connecting the velocity and the depth of the snow, and Eqs. (2.3) are written, respectively, as

$$\mathbf{v} = \sqrt{h}\, f(x)\,\mathbf{e}, \qquad \frac{dx}{dt} = \frac{3}{2} f \sqrt{h}, \qquad \frac{dh}{dt} = -\frac{h^{3/2}}{S}\frac{d(Sf)}{dx}$$

$$f(x) = \sqrt{g(\sin\psi - \mu\cos\psi)/k}$$

In this case Eqs. (2.3) can be integrated and yield

$$h = C(Sf)^{-2/3}, \quad t = \frac{2}{3\sqrt{C}} \int_{x_0}^{x} \frac{S^{1/3}}{f^{2/3}}\, dx + t_0 \tag{2.4}$$

Here C, x_0, t_0 are constants along the characteristic. Using the initial and boundary conditions we can eliminate C, x_0, t_0 from relations (2.4) and obtain the solution in the form $h = h(x, t)$, $v = f\sqrt{h} = v(x, t)$.

In particular, the region of initial perturbation, from which the avalanche develops, can in many cases be taken as small from the large-scale point of view and can be replaced within each strip between two lines of steepest descent by a point $(x_0 = 0$, $t_0 = 0)$ above which the snow remains unmoving. A beam of characteristics (a rarefaction wave) emanates from that point. If $V(h) + h\partial V / \partial h > 0$ for all $h > 0$, then along the last characteristic the condition $h = 0$ should be given. The distribution of h and v in the rarefaction wave issuing from one point, obtained by (2.4), has the form

$$h = \frac{4}{9}\frac{\varphi^2(x)}{t^2}, \qquad v = \frac{2}{3}\frac{f(x)\varphi(x)}{t}$$

$$\varphi(x) = (Sf)^{-1/3} \int_0^x \frac{S^{1/3}}{f^{2/3}}\, dx$$

3. Boundary conditions for the simplified system. Conditions on the surface of discontinuity. When constructing the large-scale solution the boundary conditions are the conditions on the narrow boundary zones of abrupt variation of the parameters of the moving snow or on the surfaces of discontinuity replacing these zones. One of the conditions on the surface of discontinuity is the condition of conservation of mass, which follows from (1.1), (1.4), (1.7), (2.1) and can be written as

$$w = [hV(h) - h_0V(h_0)] / (h - h_0) \tag{3.1}$$

$$w = h\,V(h) / (h - h_0) \tag{3.2}$$

for the internal discontinuities and for the discontinuity which simulates the narrow zone abutting the leading front of the avalanche (the break-down front), respectively. Here w is the velocity of the discontinuity along the line of steepest descent, $w = w_n / \cos\theta$, θ is the angle between the normal to the surface of discontinuity and the direction of steepest descent, h and h_0 are the snow depths behind and ahead of the discontinuity, respectively. Conditions (3.1) and (3.2) are found in correspondence with Eq. (2.2) which also expresses the conservation of mass.

If we draw a graph of the function $hV(h)$ (Fig. 1), then expressions (3.1) and (3.2) for the velocity w of the discontinuity can be treated as tangent of the slope angle of the secant which, in the first case, passes through the point with coordinates h_0, $h_0V(h_0)$ and, in the second case, passes through the point with coordinates h_0, 0. According to (2.3) the propagation rate of characteristics of the large-scale motion equals the tangent of the slope angle of the tangent to the graph of $hV(h)$. We shall assume that this quantity is a monotonically increasing function of h. Then, the graph of the function $hV(h)$ is convex downwards as shown in Fig. 1.

From Fig. 1 we see that if $h > h_0$, the inequality

$$a(h_0) \leqslant w \leqslant a(h), \quad a(h) \equiv d/dh\,[h\,V(h)] \tag{3.3}$$

is fulfilled for the internal discontinuities and equality can hold only for $h = h_0$. To each $h \geqslant h_0$ corresponds a value $w(h)$, where $w(h)$ is a monotonically increasing function. As will follow from what we say later on, solutions representing their structure do not correspond to discontinuities with $h < h_0$. Therefore, we do not consider such discontinuities. Furthermore, when $h < h_0$ the signs of the inequalities in (3.3) are reversed, and if we do not lay down (three) additional conditions, such a discontinuity does not evolve [4]. On Fig. 1 we mark the point G at which the ray drawn from point h_0, 0 is tangent to the curve $hV(h)$. We see that

$$\begin{aligned} &w_G = w(h_G) = a(h_G), \; w \geqslant w_G \\ &w < a(h), \quad h > h_G; \\ &w > a(h), \quad h < h_G \end{aligned} \tag{3.4}$$

Note that if (3.2) is the only condition on a surface of discontinuity, then the leading front is a nonevolving discontinuity for $h < h_G$. However, for certain cases we can find an additional condition which should be laid down for $h < h_G$. If from the assumption it turns out that $a(h)$ is a monotonic function, then internal discontinuities are possible with a decrease in h.

Let us consider the motion in a narrow zone corresponding to a discontinuity of the large-scale solution. The parameters of the slope inside this zone can be taken to be constant, and the motion assumed uniform and in steady state in the system of coordinates ξ, η, moving together with the leading front with velocity $w_n = w\cos\theta$. In this case Eqs. (1.1), (1.2) become

$$\begin{aligned} &h(w_n - u) = Q \\ &(u - w_n)\frac{du}{d\xi} + g\cos\psi\frac{dh}{d\xi} = g\sin\psi\cos\theta - F\left(\sqrt{u^2+v^2}, h\right)u \\ &(u - w_n)\,dv/d\xi = g\sin\psi\sin\theta - F\left(\sqrt{u^2+v^2}, h\right)v \end{aligned} \tag{3.5}$$

Here u, v are the projections of the absolute velocity onto the ξ-axis directed along the normal to the leading front downward with respect to the motion and the η-axis directed along the tangent, respectively. Q is the snow's mass flow divided by its density. In the zone abutting the break-down front, $Q = h_0 w_n$; in the zone corresponding to an internal discontinuity, $Q = h_0(w_n - u_0)$.

Eliminating h with the aid of the first equation in (3.5), we obtain a system of two equations for $u(\xi)$ and $v(\xi)$. The solution $u(\xi)$, $v(\xi)$, representing the avalanche's leading front, should start at the break-down front and end at one of the singular points of this system, where $du/d\xi = 0$, $dv/d\xi = 0$. The structure of the internal discontinuity represents the transition from one singular point to the other. The velocity componets corresponding to the singular points of system (3.5) are determined by the relations

$$v/u = \operatorname{tg}\theta \equiv \delta$$
$$u(1+\delta^2)\,F\left(u\sqrt{1+\delta^2}, \frac{Q\sqrt{1+\delta^2}}{w-u\sqrt{1+\delta^2}}\right) = g\sin\psi$$

The corresponding equation for determining the quantity h at the singular points can obviously be written in the form (3.1) or (3.2). The investigation carried out above with the aid of relations (3.1) and (3.2) shows that if $a(h)$ is a monotonically increasing function of h, then (depending on the magnitude of w) there exist no more than two singular points of system (3.5) (points A and B in Fig. 2), and

$$a(h_B) \leqslant w \leqslant a(h_A) \tag{3.6}$$

On the (u, v)-plane we mark the "critical" straight line (*)

$$u = u_{cr} \equiv w_n - \sqrt[3]{Qg\cos\psi}$$

on which $w_n - u = \sqrt{gh\cos\psi}$. Under a monotonic variation in ξ it is impossible to pass through this line continuously, because as we pass through it the derivative of u with respect to ξ generally changes sign. A passage through the critical straight line is possible only by the jump (1.4), (1.5) or (1.7) which always occurs with an increase in u and h [2]. The type of the singular points A and B can be found directly from Eqs. (3.5) or by means of the theorem proved in [5].

It is known [6] that if the motion with constant parameters satisfying both, the complete as well as the simplified systems, is stable (which we assume), then the characteristics of the simplified system cannot overtake those of the complete system. Therefore, the inequality

$$u + \sqrt{gh\cos\psi} \geqslant a\cos\theta \tag{3.7}$$

is fulfilled at the singular points. From [2] it follows that for a force of friction given by equality (1.3), inequality (3.7) is equivalent to the inequality $\operatorname{tg}\psi - \mu < 4k / \cos^2\theta$. From (3.6) and (3.7) we see that the singular point A always lies to the right of the critical straight line $u = u_{cr}$ and is a node or a focus with integral curves entering into it as $\xi \to -\infty$ (see Fig.2). The type of the singular point B depends upon its position relative to the straight line $u = u_{cr}$. If it lies to the left of this straight line, i.e. if

$$w_n > u_B + \sqrt{gh_B\cos\psi} \tag{3.8}$$

then point B is a node or a focus with entering integral curves; however, if A and B are located to one side of the straight line $u = u_{cr}$, i.e. if inequality (3.8) is of opposite sign, then point B is a saddle. For internal discontinuities there always exists a solution of the complete system, starting at point B and ending at point A, describing their structure. When inequality (3.8) is fulfilled, this solution starts with a jump determined by conditions (1.7), but is continuous when inequality (3.8) has the opposite sign.

Let us now consider in more detail the structure of the avalanche's leading front. The corresponding solution of system (3.5) starts at the break-down front and ends at point A or at point B. In the first case we say that the leading front is of type A, while in the second, of type B. From relations (1.4), (1.5) we see that immediately behind the

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*) Editor's Note. Here and in the sequel the subscript " cr " denotes "critical".

break-down front the point (u, v) lies on the u-axis (point C on Fig. 2), and under condition (1.6) we always have

$$w_n < u_C + \sqrt{gh_C \cos\psi}, \quad u_C > u_{cr} \tag{3.9}$$

If $u_B < u_{cr}$, then by virtue of (3.9) the desired solution is depicted by an integral curve going into point A. The transition to point B would be accompanied by a passage through the critical straight lines with a decrease in u and h, which is impossible. Let point B be a saddle. We consider the integral curve entering into point B. On this integral curve there can exist a point (point D on Fig. 2) at which $v = 0$. If there is no such point, then the solution describing the structure of the discontinuity being examined can terminate only at the point A. If point D exists, then for $u_C > u_D$ the solution ends at point A, for $u_C < u_D$ the solution representing the structure of the avalanche's leading front does not exist, and for $u_C = u_D$ the solution ends at point B. The condition

$$u_C = u_D \tag{3.10}$$

is an additional condition which should be fulfilled at discontinuities of type B. The quantities u_C and u_D can be expressed in terms of the front's velocity w and of the parameters of the slope and the snow. Therefore, condition (3.10) can be looked upon as an equation for w. Its solution, if it exists, can be written as a function of the snow and slope parameters at the spot the front is located and of the quantity δ characterizing the front's angle of inclination to the line of steepest descent

$$w = w^* (\psi, \ldots, \delta) \tag{3.11}$$

The problem of finding relation (3.11) in explicit form, including the determination of integral curve DB, can be solved, it seems, only by numerical methods. In general, the function w^* need not be single-valued. From the region of admissible values of w, which is defined by the inequality $w > w_G$, the values of w^* delineate segments such that to the values of w belonging to these segments there does not correspond a solution representing the leading front's structure. To values of w not belonging to these segments there correspond fronts of type A, while to the endpoints of the segments, fronts of type B. In those cases when there is no point D or when Eq. (3.10) is not fulfilled for any values of w whatsoever, all values $w \geqslant w_G$ correspond to discontinuities of type A.

When the force of friction is given by equality (1.3), for $\delta = 0$ the difference $u_C - u_D$ is a monotonically increasing function of w. If the monotony of the dependency of $u_C - u_D$ on w is preserved in some region of values of δ, then no more than one value of w^* exists. If such a value does not exist, then the discontinuity is of type A for all $w > w_G$. However, if w^* exists, then to values $w < w^*$ there do not correspond discontinuities possessing a structure, to the value $w = w^*$ there corresponds a discontinuity of type B, and to values $w > w^*$, a discontinuity of type A. Here, the obvious inequality $w^* \geqslant w_G$ is satisfied in the region wherein discontinuities of type B are possible, in the space of the variables characterizing the state of the snow and slope and the direction of the shock wave, and the boubdary of this region is given by the equality

$$w^* = w_G \tag{3.12}$$

The condition for the existence of waves of type B has been found in explicit form [2]

$$\sigma^* < \rho g h_0 \left(1 - \frac{\operatorname{tg}\psi - \mu}{k}\right) \cos\psi$$

and the function w^* has been investigated qualitatively, for the case when $\delta = 0$ and the force of friction is determined by formula (1.3).

In this paper we assume that the magnitude of δ is not large in those cases when we require an explicit expression for w^*. From symmetry it follows that the dependency of w^* on δ is even; therefore, by expanding the function w^* in a power series in δ^2 and restricting ourselves to two terms, we obtain

$$w^* = w_0 + K\delta^2 \tag{3.13}$$

4. Splitting up of discontinuities. It is important to note the following fact. When $w > w^*$ the integral curve representing the structure of discontinuity arrives at point A. When $w = w^*$ this integral curve turns into the integral curves DB and BA, the first of which represents the structure of a discontinuity of type B, while the second, the structure of an internal discontinuity. Thus, a discontinuity of type A splits up into a discontinuity of type B and an internal discontinuity moving with the same velocity when $w = w^*$. It seems that the splitting up of discontinuities is a rather typical behavior of discontinuities in continuum mechanics (for example, see [7, 8]).

To illustrate the process of splitting of a discontinuity we consider the "piston" problem for Eq. (2.2). We assume that the motion of the snow on a slope with constant parameters is supported from behind by the motion of a piston on which we are given $h = h_p = \text{const}$ (*). For large h_p the solution is represented by a discontinuity of type A and by a flow with homogeneous parameters following behind it. With a decrease in h_p the velocity w of the discontinuity drops in correspondence with (3.2), and if the slope is such that the discontinuities of type B can exist, then for some $h_p = h_A(w^*)$ corresponding to $w = w^*$ the solution has the same form but, as follows from the investigation of the structure, the surface of discontinuity is two discontinuities: a leading front of type B and an internal discontinuity, one moving behind the other with the same velocity. Under a further decrease in h_p the wave of type B retains its velocity, equal to w^*, while the velocity of the internal discontinuity decreases in amplitude with it. When $h_p = h\ (w^*)$ the amplitude of the internal discontinuity vanishes and under a further decrease in h_p a rarefaction wave appears in the solution. Finally, when $h_p = 0$ the solution acquires the form found in [2].

5. Motion of the avalanche when its leading front is a discontinuity of type A. If a discontinuity is of type A, the velocity w of its motion along a line of steepest descent depends, according to (2.1), on the value of h immediately behind the discontinuity, which in its own turn is determined by integrating the Eqs. (2.3) of the characteristics along the line of steepest descent. Thus, in this case, the motion of the leading front is determined along each line independently of the rest. The form of the leading front ov the avalanche can be found by joining the points specifying the position of the discontinuity on each line of steepest descent.

When the h behind the discontinuity is greater than h_G, the characteristics catch up

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*) Note. The subscript "p" stands for "piston".

with the discontinuity (Fig. 3) and its velocity is determined by h. If the value of h yielded by characteristics at the discontinuity decreases and becomes equal to h_G (at point C in Fig. 3), then the velocity of the discontinuity becomes equal to the velocity of the characteristic. If along a characteristic running in the immediate vicinity of the shock wave the quantity $h - h_G$ continues, after becoming zero, to decrease by virtue of Eqs. (2.3) and of the giving of $h_0(x)$, then this characteristic lags behind the discontinuity. In fact, according to (3.3) the velocity of the discontinuity cannot be less than that of the characteristic computed from the magnitude h_G. In this case, as long as

$$\frac{d}{dx}(h - h_G) < 0 \tag{5.1}$$

the condition $w = w_G$, analogous to the Jouguet condition, is satisfied at the discontinuity. On the (x, t)-plane the characteristics (2.3), touching, leave the surface of discontinuity. If the sign in inequality (5.1) is replaced by the opposite one (point D in Fig. 3), then the characteristics, which not long before this lagged behind the shock wave, begin to catch up with it, bringing along with them the values of h greater than h_G.

There is no difficulty in writing out the first-order ordinary differential equation describing the motion of the avalanche's leading front on segments BC and DF. When studying the motion of the avalanche and of its leading front we must keep in mind the possibility of the formation of jumps within the avalanche.

6. Motion of discontinuities under the possibility of a split. We first consider the case $\delta = 0$. Let the initial data be such that a discontinuity of type A is realized for small t (Fig. 4). If during the advance of the discontinuity its velocity becomes equal to w^*, then the discontinuity splits up into a discontinuity of type B and an internal discontinuity. The velocity w^* of motion of a discontinuity of type B is determined solely by the properties of the slope and of the snow at the point where it is located. The characteristics emanating from the discontinuity determine the flow of snow in the zone adjacent to the discontinuity and, also, affect the motion of the internal discontinuity. A discontinuity of type B can turn once again into a discontinuity of type A if the slope becomes such that a wave of type B cannot exist on it or if it catches up with an internal discontinuity whose velocity is determined by its height and is greater than the velocity of the characteristic (Fig. 4).

Let us now consider the motion of the avalanche's leading front when it is represented by a discontinuity of type B with $\delta \neq 0$. We introduce a curvilinear coordinate system with coordinate lines x directed along the lines of steepest descent and with coordinate lines y orthogonal to them. Let the length of the vector $\{dx, dy\}$ be expressed by the equality

$$ds^2 = g_1^2(x, y)\,dx^2 + g_2^2(x, y)\,dy^2 \tag{6.1}$$

We specify the law of motion of the discontinuity by the equation $x = X(y, t)$. Then the velocity w of the discontinuity and the tangent δ of the inclination to the line of steepest descent are expressed as follows

$$w = g_1 \frac{\partial X}{\partial t}, \qquad \delta = \frac{g_1}{g_2}\frac{\partial X}{\partial y} \tag{6.2}$$

Substituting these expressions into equality (3.13) which expresses the velocity of motion of the discontinuity, we obtain a first-order partial differential equation in X

$$g_1 \frac{\partial X}{\partial t} = w^* \left(X, y, \left(\frac{\partial X}{\partial y} \right)^2 \right) \tag{6.3}$$

The dependence of w^* on x and y is the result of the dependence of ψ, σ^*, F, g_1, g_2 on x and y. In (6.3) we have taken into account that w is an even function of δ. The characteristics of Eq. (6.3) are inclined to the lines of steepest descent and that implies the possibility of the motion transferring from some lines of steepest descent to others not initially taken up by motion of the avalanche.

For simplicity let us consider the motion of a front of type B on a uniform slope. Then w^* does not depend on x and y explicitly, while the dependency on δ is of form (3.13). We first find the form of the leading front of the avalanche, arising from a point perturbation at the origin. Writing the equations of the characteristics for Eq. (6.3), we find

$$\frac{dX}{dt} = w_0 - K \left(\frac{\partial X}{\partial y} \right)^2 \tag{6.4}$$

$$\frac{dy}{dt} = -2K \left(\frac{\partial X}{\partial y} \right), \quad \frac{\partial X}{\partial y} = \text{const}$$

Integrating these equations and eliminating $\partial X / \partial y$, we obtain the equation for the front

$$\frac{X}{t} = w_0 - \frac{1}{4K} \frac{y^2}{t^2} \tag{6.5}$$

Thus, the form of the avalanche's leading front developing from a point perturbation represents a parabola expanding in all directions proportionally with time. For $K > 0$ the parabola is reverse with its vertex downward relative to the slope. The width of the avalanche is inversely proportional to $\sqrt{K}$.

We now consider the behavior of the perturbation of the front's form which for $t = 0$ is represented by the straight line $X = \alpha y$ everywhere except for a certain segment of a curve in the neighborhood of the point $x = 0$, $y = 0$. For definiteness we consider the case $K > 0$ (the case $K < 0$ is investigated analogously). Let the perturbation of the front be a convexity facing downward (Fig. 5). On the perturbation there is a point at which $\partial X / \partial y = \alpha$. The normal comonent at this point of the characteristic's velocity coincides with the front's velocity and, therefore, this element of the front moves at a constant distance ahead of the main front. The y-axis component of the characteristic's velocity, corresponding to this element, is directed to the left if $\alpha < 0$ and to the right if $\alpha > 0$. The remaining characteristics corresponding to the perturbation diverge from this characteristic to different sides so that the perturbation propagates. After the lapse of a sufficiently large time the perturbation of the front represents a part of the parabola

$$X = w_0 t + C - \frac{1}{4K} \frac{y}{t} \tag{6.6}$$

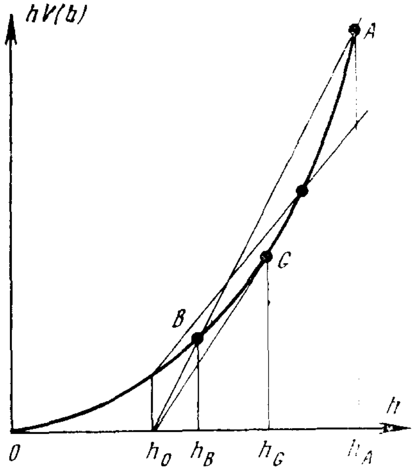

Fig. 1

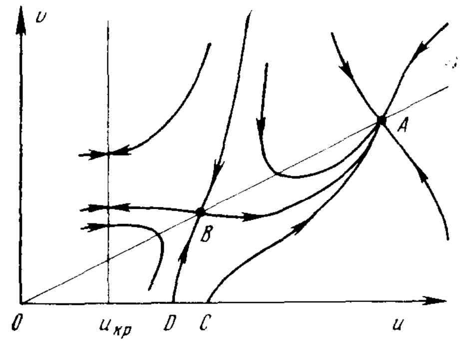

Fig. 2

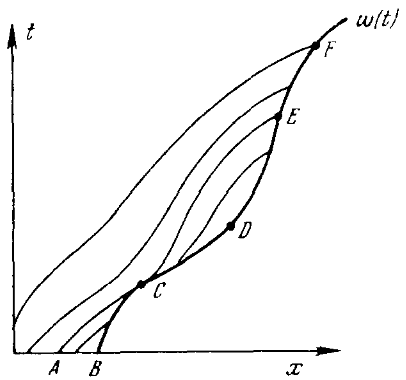

Fig. 3

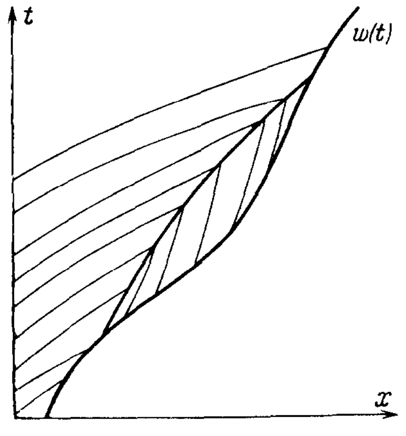

Fig. 4

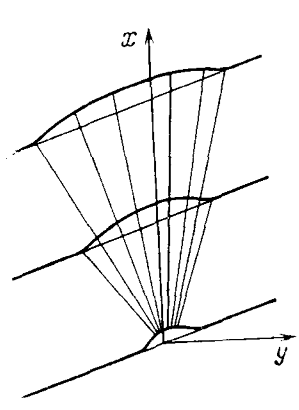

Fig. 5

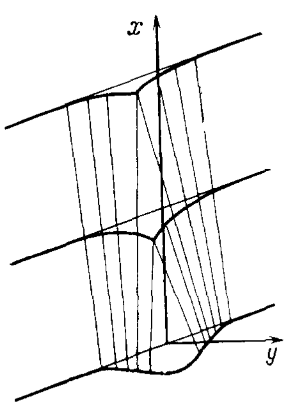

Fig. 6

(C is the amplitude of the initial perturbation) which lies below the straight line

$$X = \alpha y + (w_0 + K\alpha^2)t \tag{6.7}$$

representing the remaining part of the front. There is a discontinuity in the derivative $\partial X / \partial y$ at the place where the parabola (6.6) intersects the straight line (6.7). For large t the width of the perturbation grows as $\sqrt{t}$.

When the perturbation represents a concavity in the front, the characteristics corresponding to the different point of the perturbation converge (Fig. 6). Where they intersect a discontinuity is formed in the derivative $\partial X / \partial y$. The characteristics corresponding to the middle part of the initial perturbation terminate on the discontinuity. Therefore, the perturbation's amplitude tends to zero with increasing time.

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